On the exact WKB analysis for operators admitting infnitely many phases (Asymptotic Analysis and Microlocal Analysis of PDE)

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On the exact WKB analysis for operators admitting infinitely many phases

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1 Local theory near a simple turning point

Let us consider a formal differential operator of infinite order in one variable x with a large parameter η that has the following form:

$$P(x,\partial_x,\eta)=P(x,\partial_x/\eta)=\sum_{j,k=0}^{\infty}a_{j,k}x^j\eta^{-k}\partial_x{}^k, \tag{1}$$

where $a_{j,k}$ are complex constants. We assume that its Borel transform

$$P_B=P(x,\partial_x/\partial_y)=\sum_{j,k=0}^{\infty}a_{j,k}x^j\partial_x{}^k\partial_y{}^{-k} \tag{2}$$

is a microdifferential operator of order 0 defined outside $\{\eta=0\}$; that is, we assume that the symbol $P(x,\xi/\eta)$ of P_B is an entire function of $\zeta=\xi/\eta$ which depends holomorphically on x. Otherwise stated, $P(x,\zeta)$ is a holomorphic function in $U\times\mathbb{C}$, where U is an open set in $\mathbb{C}$.

Suppose that the system of equations

$$P(x,\zeta)=\partial_\zeta P(x,\zeta)=0 \tag{3}$$

has a solution $(x,\zeta)=(x_*,\zeta_*)$ $(x_*\in U)$. Then x_* is said to be a turning point of the operator P with the characteristic value ζ_*. Let x_* be a turning point with the characteristic value ζ_*. We say that x_* is simple if

$$\partial_x P(x_*,\zeta_*)\neq 0 \quad\text{and}\quad \partial_\zeta^2 P(x_*,\zeta_*)\neq 0. \tag{4}$$

Near a simple turning point x_* we can find two multivalued analytic functions $\zeta_\pm(x)$ of the equation $P(x,\zeta)=0$ for which

$$\zeta_\pm(x_*)=\zeta_* \quad\text{and}\quad \zeta_+(x)-\zeta_-(x)=\sqrt{x-x_*}\,f(x) \tag{5}$$

hold with a holomorphic function $f(x)$ that does not vanish at $x=x_*$, and we further find that $\zeta(x_*)\neq\zeta_*$ for any other analytic solution $\zeta(x)$ of $P(x,\zeta)=0$. We also note that $\zeta_+(x)+\zeta_-(x)$ and $\zeta_+(x)\zeta_-(x)$ are holomorphic functions in a neighborhood of x_*.

Theorem 1 Suppose that for every $x\in U$, there is a solution $\zeta=\zeta(x)$ of the equation $P(x,\zeta)=0$ and $(\partial P/\partial\zeta)(x,\zeta(x))$ is not identically zero. Then there exists a formal solution ψ of the differential equation

$$P(x,\partial_x,\eta)\psi=0 \tag{6}$$

that is of the form

$$\psi = \exp(\int S(x,\eta)dx) \tag{7}$$

with

$$S(x,\eta) = \sum_{j\geq -1} \eta^{-j} S_j(x) \quad \text{and} \quad S_{-1}(x) = \zeta(x). \tag{8}$$

Here S is unique up to the choice of a branch of $\zeta(x)$.

A solution (7) is called a WKB solution of (6).

Theorem 2 Let x_* be a simple turning point with the characteristic value ζ_* and let $\zeta_\pm(x)$ be solutions of $P(x,\zeta) = 0$ satisfying (5). Let $r(x,\zeta)$ denote a quadratic polynomial $(\zeta - \zeta_+(x))(\zeta - \zeta_-(x))$ with holomorphic coefficients in a small neighborhood V of x_*. Then there exists a differential operator of infinite order with the large parameter η

$$Q(x,\partial_x/\eta,\eta) = \sum_{j=0}^{\infty} \eta^{-j} Q_j(x,\partial_x/\eta) \tag{9}$$

and a second-order differential operator with the large parameter η

$$R(x,\partial_x/\eta,\eta) = \sum_{j=0}^{\infty} \eta^{-j} R_j(x,\partial_x/\eta) \tag{10}$$

so that the following conditions are satisfied:
(i) $Q_j(x,\zeta)$ and $R_j(x,\zeta)$ are holomorphic in $V \times \mathbb{C}$ and $Q(x,\partial_x/\partial_y,\partial_y)$, $R(x,\partial_x/\partial_y,\partial_y)$ are microdifferential operators of order 0.
(ii) $P(x,\partial_x/\eta) = Q(x,\partial_x/\eta,\eta)R(x,\partial_x/\eta,\eta)$.
(iii) $R_0(x,\zeta) = r(x,\zeta)$.
(iv) For each $j > 0$, $R_j(x,\zeta)$ is of degree at most one in ζ.

Theorem 2 above implies that, near a simple turning point x_*, WKB solutions of (6) with the leading term $S_{-1}(x) = \zeta_\pm(x)$ satisfy a second-order differential equation

$$R(x,\partial_x/\eta,\eta)\psi = 0. \tag{11}$$

Hence Stokes curves emanating from x_* should be defined by

$$\mathrm{Im} \int_{x_*}^{x} (\zeta_+(s) - \zeta_-(s))ds = 0, \tag{12}$$

and a local connection formula for WKB solutions can be obtained by reducing the problem to the second order case.

2 Study of the global configuration of Stokes curves — an example

To illustrate the general theory discussed in §1 and to try to find what occurs globally, we consider the following example:

$$P(x,\partial_x/\eta)\psi = 0, \quad P(x,\partial_x/\eta) = \cosh(\sqrt{\partial_x/i\eta}) - x. \tag{13}$$

Turning points for (13) are $x = 1$ and $x = -1$, and they are simple in the sense of (4). The leading term $S_{-1}(x)$ of a WKB solution is given as follows:

$$S_{-1} = f_n, \quad f_n(x) = i\left(2n\pi i + \log(x + \sqrt{x^2-1})\right)^2 \quad (n \in \mathbb{Z}). \tag{14}$$

Hence there are infinitely many phases for (13) and Stokes curves emanating from $x = 1$ (resp. $x = -1$) are defined by

$$\mathrm{Im}\int_1^x (f_n(s) - f_{-n}(s))ds = 0 \quad (\text{resp. } \mathrm{Im}\int_{-1}^x (f_n(s) - f_{-n-1}(s))ds = 0). \tag{15}$$

Their configuration is shown in Fig.1; note that all these Stokes curves sit on the same curves.

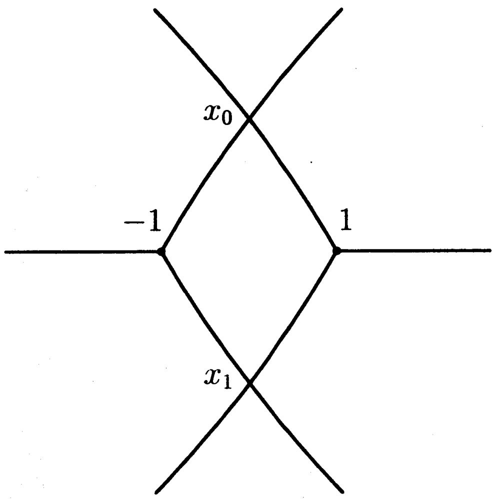


Fig.1

Since Stokes curves in Fig.1 have two crossing points x_0 and x_1, we need to introduce new Stokes curves to obtain the complete description of Stokes regions, i.e., the regions in which WKB solutions become Borel summable (cf. [BNR], [AKT1]).

Let us determine new Stokes curves passing through the crossing point x_0. To make the argument explicit we place cuts along $\{x; x \geq 1\}$ and $\{x; x \leq -1\}$ in x-plane, and fix a branch of $\log(x + \sqrt{x^2-1})$ so that

$$\sqrt{x^2-1}\Big|_{x=0} = i \quad \text{and} \quad \log(x+\sqrt{x^2-1})\Big|_{x=0} = \frac{\pi}{2}i \tag{16}$$

hold. Then we can verify that ψ_n is dominant over ψ_{-n} along the Stokes curve which emanates from $x = 1$ and passes through $x = x_0$; that is,

$$\mathrm{Re}\int_1^x (f_n(s) - f_{-n}(s))ds > 0 \quad (n = 1, 2, \ldots) \tag{17}$$

holds along this Stokes curve. (In what follows, this dominance relation (17) is denoted by "$n > -n$".) We also find "$-n-1 > n$" $(n > 0)$ holds along the Stokes curve which emanates from $x = -1$ and passes through $x = x_0$. Thus we obtain the following dominance relation at $x = x_0$:

$$\text{“}\cdots > 3 > -3 > 2 > -2 > 1 > -1 > 0\text{”}. \tag{18}$$

This suggests that new Stokes curves should be given by

$$\operatorname{Im}\int_{x_0}^{x}(f_m(x) - f_n(x))dx = 0 \quad (m, n \in \mathbb{Z}), \tag{19}$$

or equivalently,

$$\operatorname{Im}\int_{x_0}^{x}(k\pi i + \log(x + \sqrt{x^2 - 1}))dx = 0 \quad (k \in \mathbb{Z}). \tag{20}$$

Among them $m = -n$ and $m = -n-1$ (i.e., $k = 0, -1$) are ordinary Stokes curves. These curves (with $-4 \leq k \leq 3$) are shown in Fig.2.

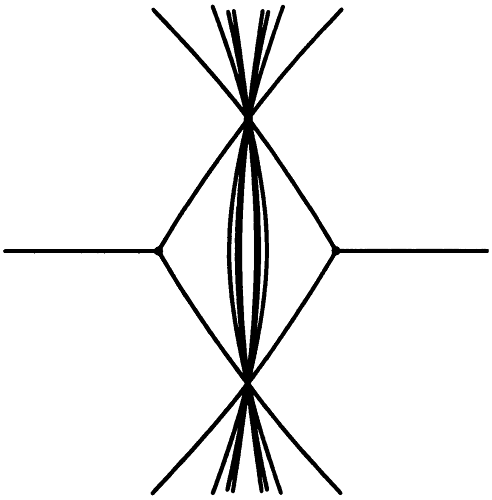


Fig.2

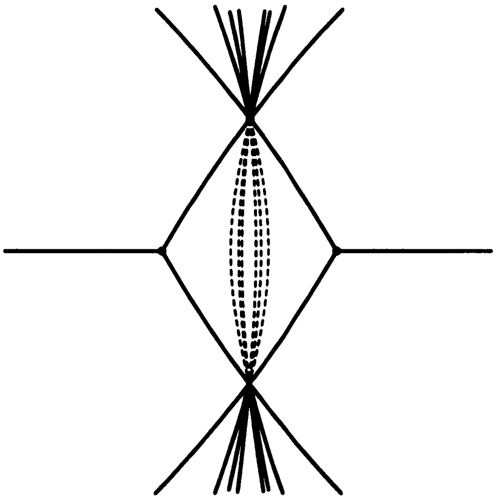


Fig.3

As [AKT1] observed, each new Stokes curve is associated with (a family of) new turning points. In our case, we can readily locate new turning points by explicitly writing down the equations for bicharacteristic strips for P_B; they are given by $x = \pm\cos w$, where w should satisfy $w = \tan w$. All solutions of this equation $w = \tan w$ are real and symmetric with respect to the origin. Let $w_1 < w_2 < \cdots$ be positive solutions of this equation. Then by numerical calculation we can check that a new Stokes curve defined in (20) passes through a new turning point $x = \cos w_k$ for $k > 0$ or $x = -\cos w_{-k-1}$ for $k < -1$. Furthermore no Stokes phenomena are expected to occur near a new turning point (cf. [AKT1, pp.77]). In Fig.3 we use a dotted line to denote the portion of a new Stokes curve along which no Stokes phenomena occur.

The above description of the Stokes geometry for the operator P is a kind of ansatz. We validate it by using the steepest descent method to the integral representation

$$\psi(x, \eta) = \int \exp(\eta h(x, \xi))d\xi \tag{21}$$

of solutions of (13) (cf. [T], [AKT2]), where

$$h(x,\xi) = x\xi - 2i\sqrt{\xi/i}\sinh(\sqrt{\xi/i}) + 2i\cosh(\sqrt{\xi/i}). \tag{22}$$

Let us first consider the configuration of steepest descent paths near $x = 1$ (notice that x is included as a parameter in the integrand of (21)). By its definition, saddle points of (21) are give by $\xi = \xi_n(x)$, where

$$\xi_n(x) = i(2\pi i + \log(x + \sqrt{x^2-1}))^2, \tag{23}$$

and our interest is in the steepest descent path C_n of $\mathrm{Re}\, h(x,\xi)$ which passes through $\xi = \xi_n(x)$. Fig.4-k shows the configuration of such saddle points and steepest descent paths for

$$x = 1 + 0.2\exp(0.1i\pi k) \quad (k = 0,1,2,\ldots,20). \tag{24}$$

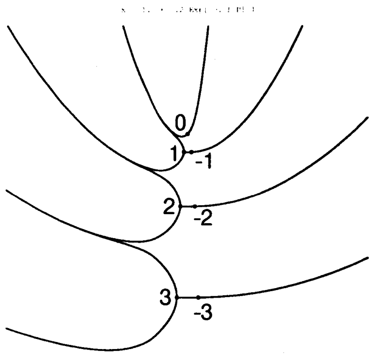


Fig.4-0

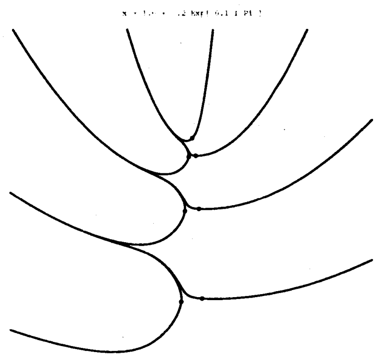


Fig.4-1

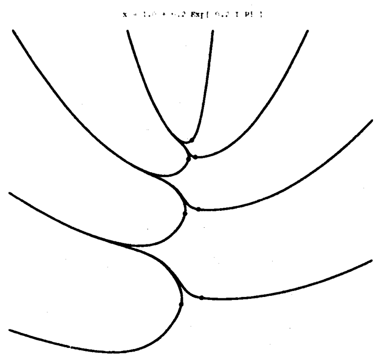


Fig.4-2

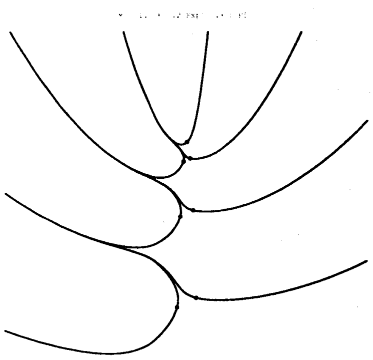


Fig.4-3

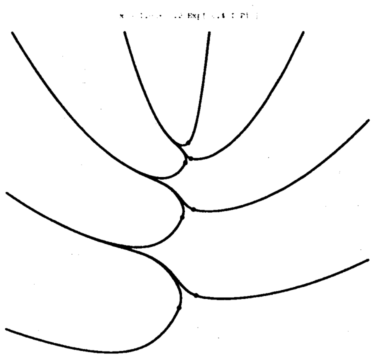
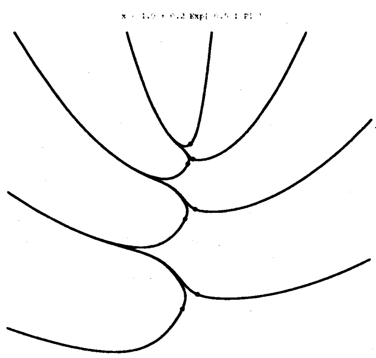


Fig.4-4



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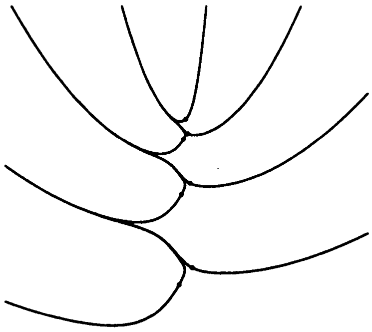
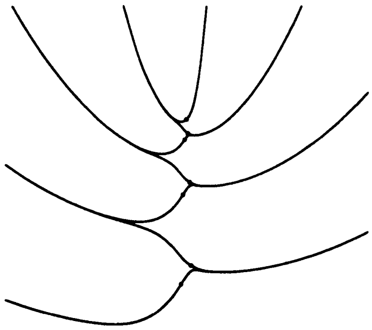


Fig.4-6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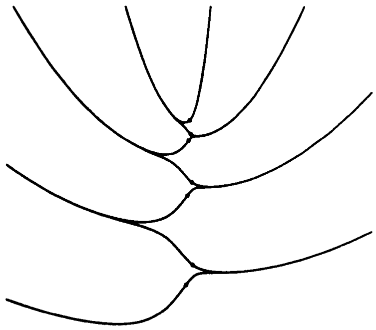
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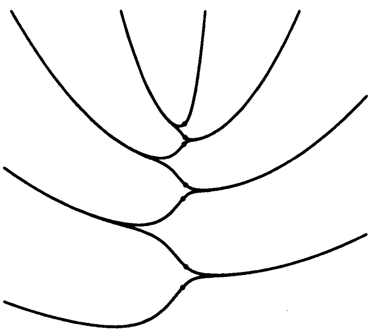
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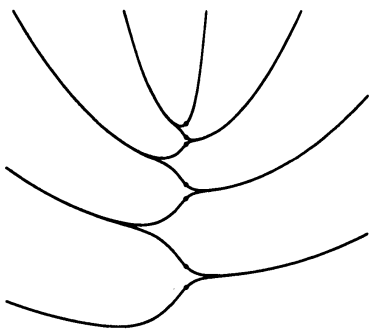
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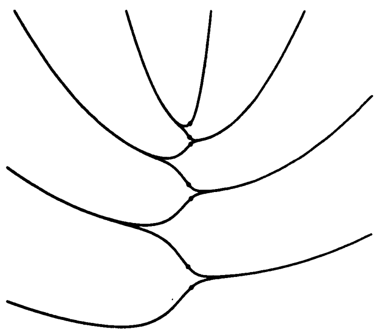


Fig.4-11

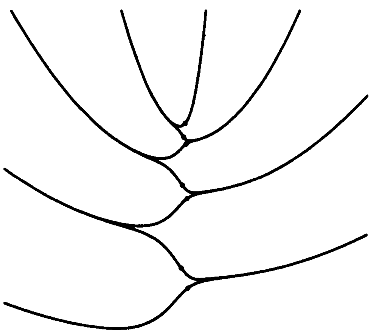


Fig.4-12

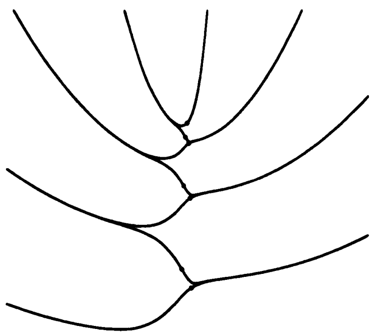


Fig.4-13

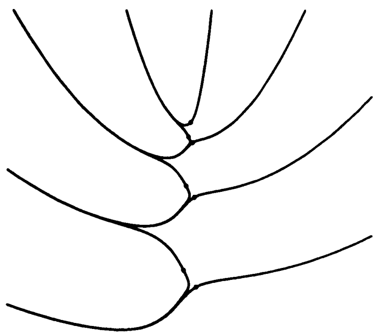


Fig.4-14

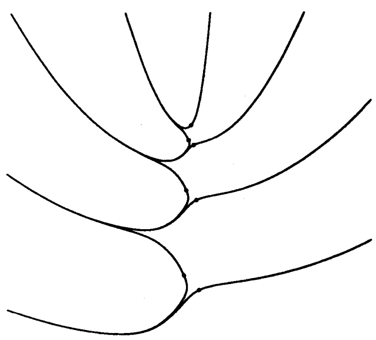
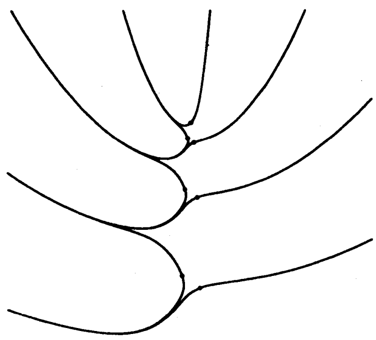


Fig.4-15



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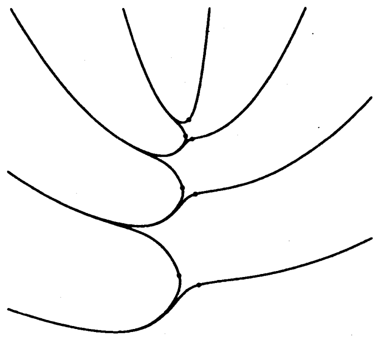

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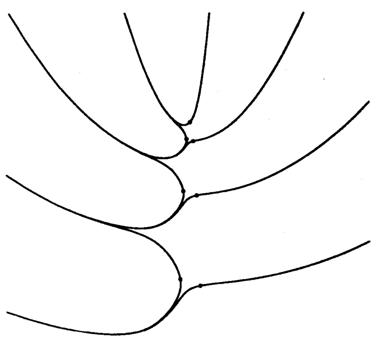
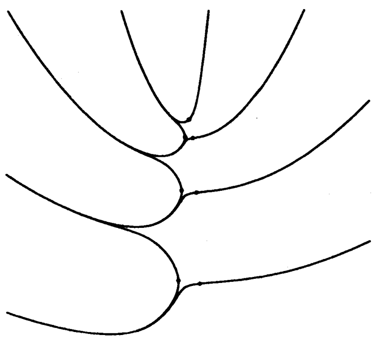


Fig.4-18



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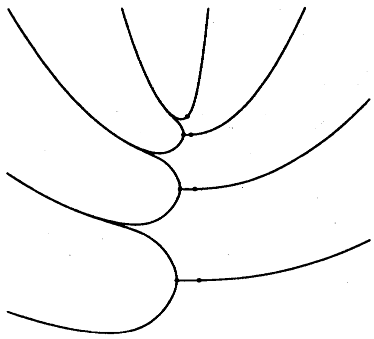

Fig.4-20

We can observe from these figures that a topological change of their configuration occurs at three places; between Fig.4-6 and Fig.4-7, between Fig.4-13 and Fig.4-14, and near Fig.4-20 (i.e., Fig.4-0). This corresponds to the fact that three Stokes curves emanate from the turning point $x = 1$; each change occurs exactly when we cross a Stokes curve emanating from $x = 1$. This implies that Fig.3 describes the Stokes geometry for (13) correctly, at least, near $x = 1$. In a similar manner we can confirm that Fig.3 describes the Stokes geometry correctly also near $x = -1$.

Next we consider the configuration of steepest descent paths when x runs parallel with the real axis. Fig.5-k shows the configuration for

$$x = (-1.5 + 0.2i) + 0.2k \quad (k = 0, 1, 2, \ldots, 14). \tag{25}$$

The configuration of steepest descent paths changes topologically at two places (between Fig.5-3 and Fig5-4, between Fig.5-11 and Fig.5-12), i.e., exactly when we cross a Stokes curve emanating from $x = 1$ or $x = -1$. In particular, no topological changes of the configuration occur when we cross the portion of new Stokes curves designated by dotted lines in Fig.3.

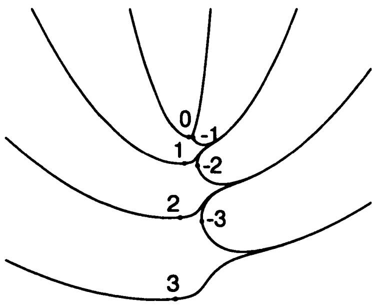

Fig.5-0

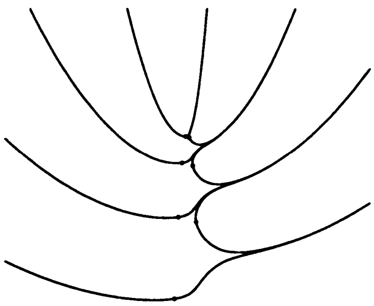

Fig.5-1

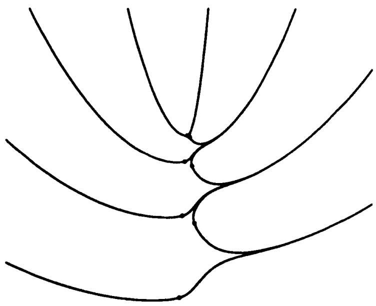

Fig.5-2

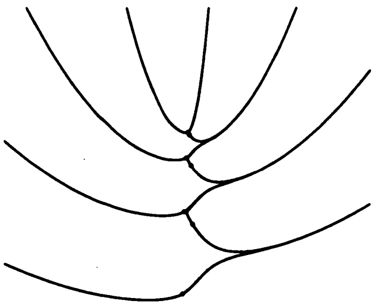

Fig.5-3

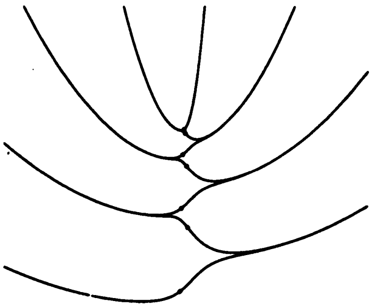

Fig.5-4

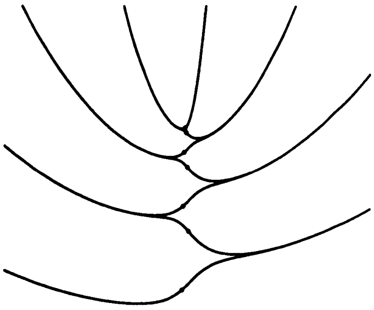

Fig.5-5

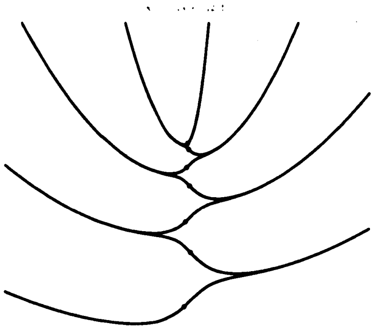

Fig.5-6

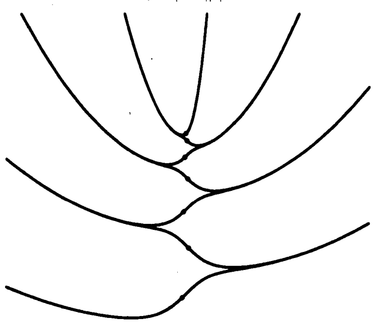

Fig.5-7

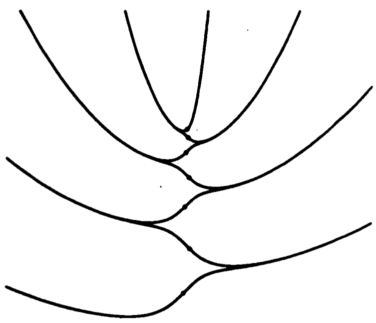

Fig.5-8

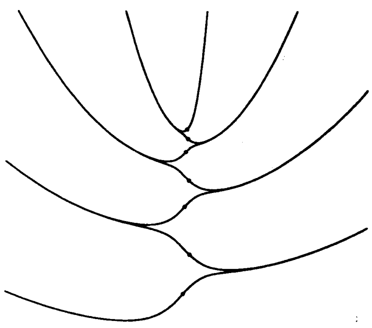

Fig.5-9

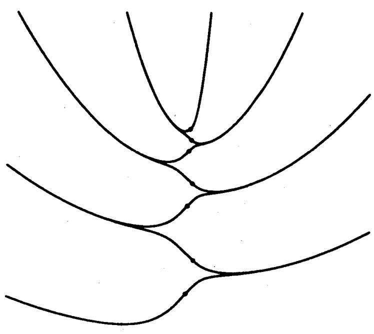

Fig.5-10

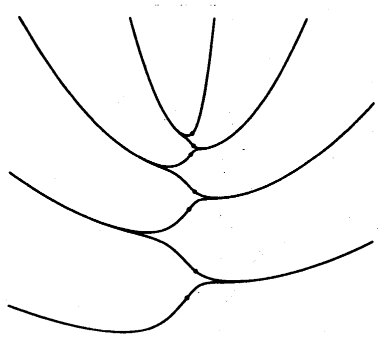

Fig.5-11

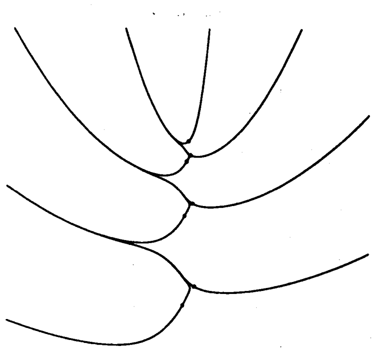

Fig.5-12

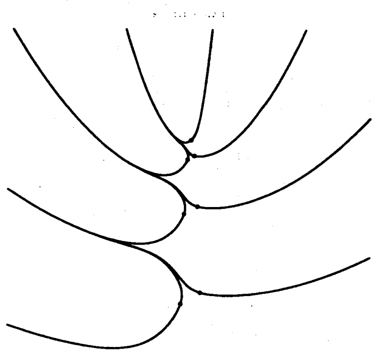

Fig.5-13

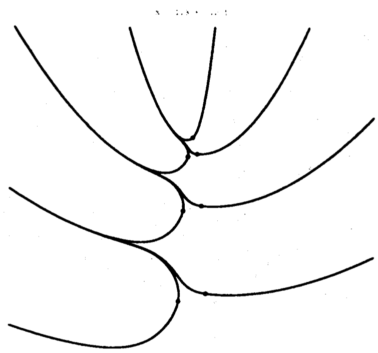

Fig.5-14

Let us now consider the configuration of steepest descent paths near $x = x_1$, a crossing point of Stokes curves in the lower half plane (cf. Fig.1). Fig.6-k shows the configuration of steepest descent paths for

$$x = x_1 + 0.2 \exp(0.1 i\pi k) \quad (k = 0, 1, 2, \ldots, 20). \tag{26}$$

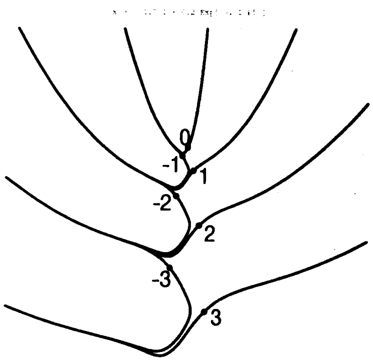

Fig.6-0

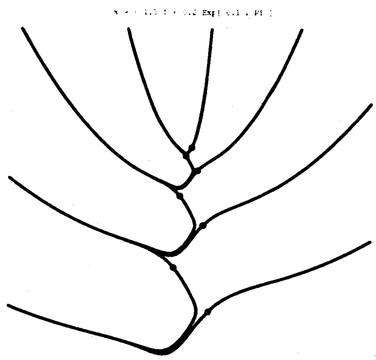

Fig.6-1

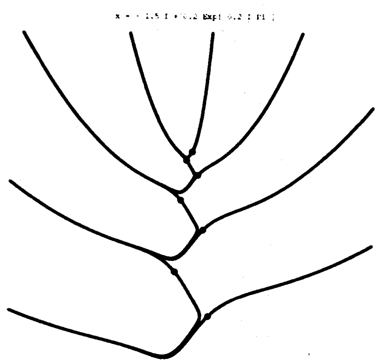

Fig.6-2

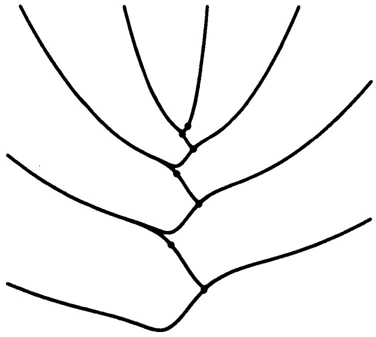

Fig.6-3

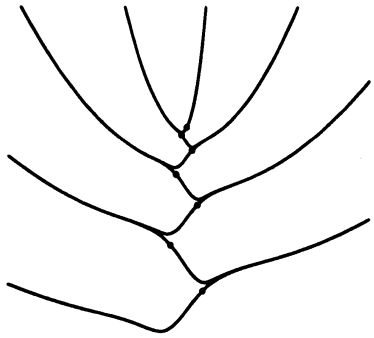

Fig.6-4

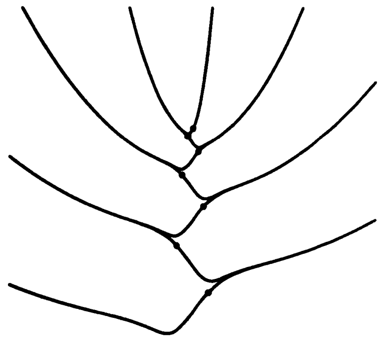

Fig.6-5

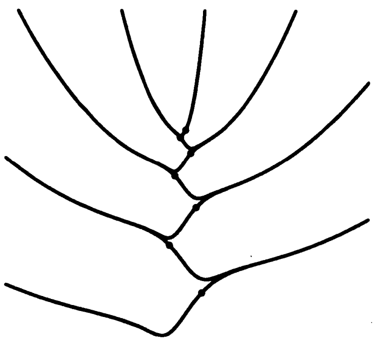

Fig.6-6

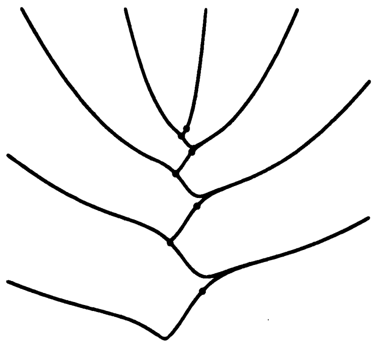

Fig.6-7

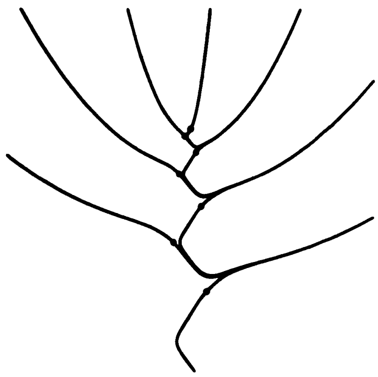

Fig.6-8

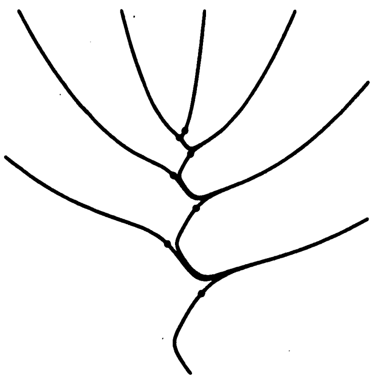

Fig.6-9

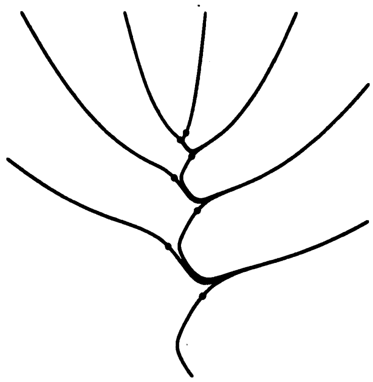

Fig.6-10

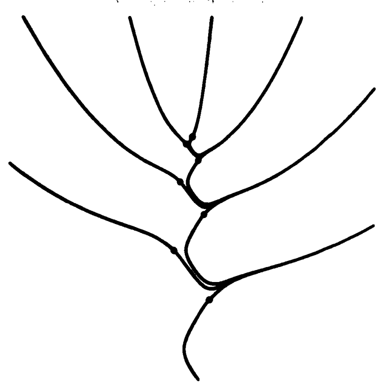

Fig.6-11

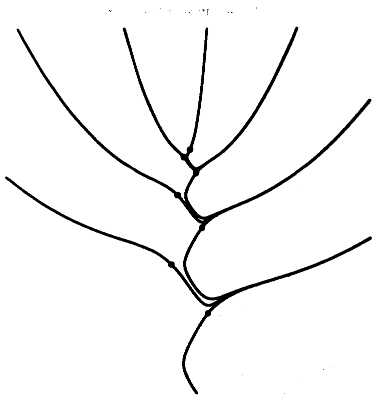
Fig.6-12

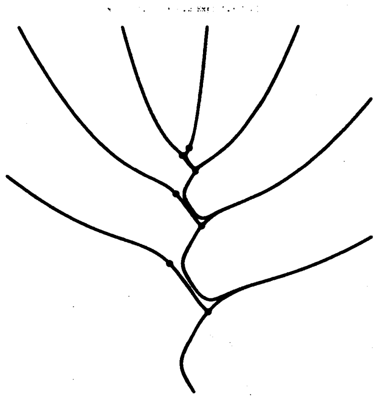
Fig.6-13

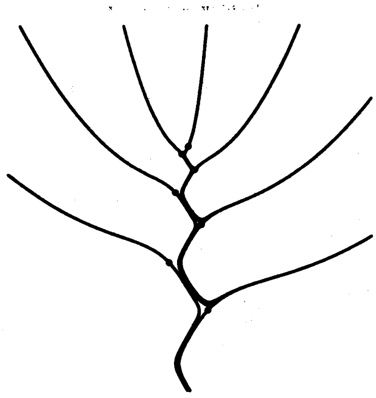
Fig.6-14

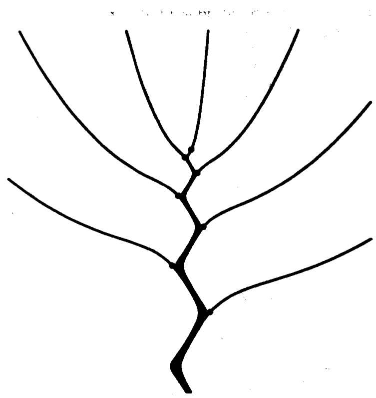
Fig.6-15

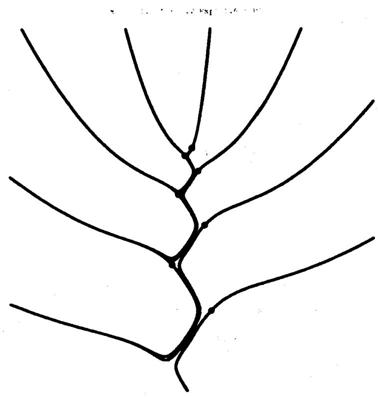
Fig.6-16

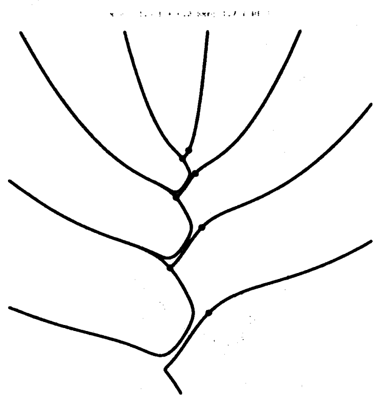
Fig.6-17

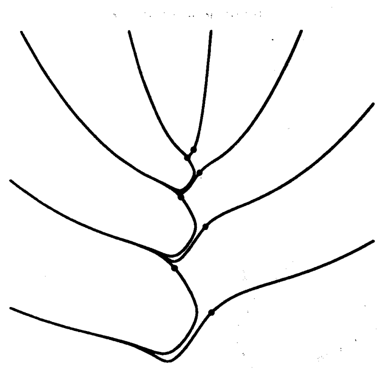
Fig.6-18

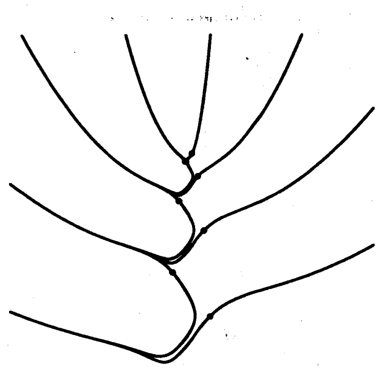
Fig.6-19

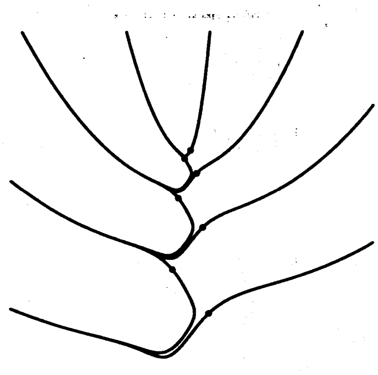
Fig.6-20

From these figures, we first find that the configuration of steepest descent paths changes topologically when we cross a Stokes curve emanating from $x = 1$ (between Fig.6-2 and Fig.6-3, between Fig.6-13 and Fig.6-14) and that from $x = -1$ (between Fig.6-7 and Fig.6-8, between Fig.6-16 and Fig.6-17). We can also verify that no topological changes occur when we cross new Stokes curves designated by a dotted line in Fig.3. In contrast with it, topological changes of the configuration can be observed when we cross new Stokes curves designated by a *solid* line in Fig.3 (from Fig.6-15 to Fig.6-17). These changes can be

visualized clearly by more precise numerical calculations; Fig.7-k shows the configura
of steepest descent paths for

$$x = x_1 + 0.2 \exp\left(i\pi(1.25 + 0.01k)\right) \quad (k = 0, 1, 2, \ldots, 25).$$

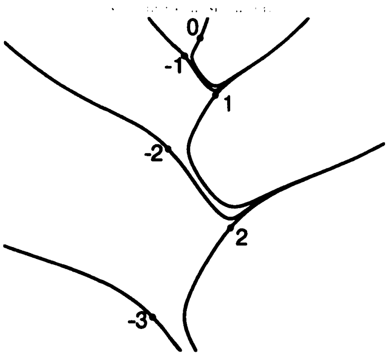

Fig.7-0

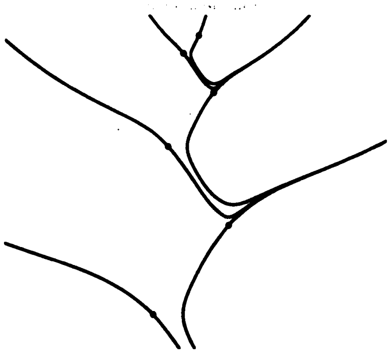

Fig.7-1

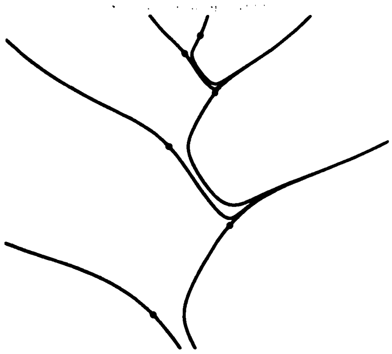

Fig.7-2

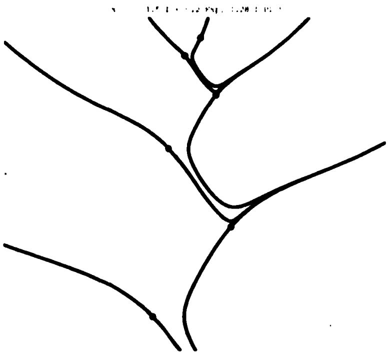

Fig.7-3

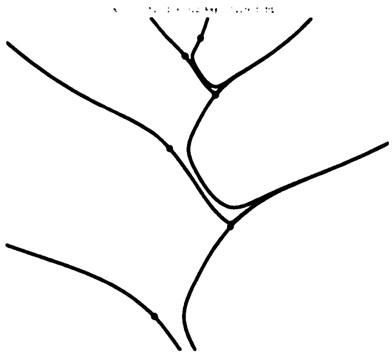

Fig.7-4

Fig.7-5

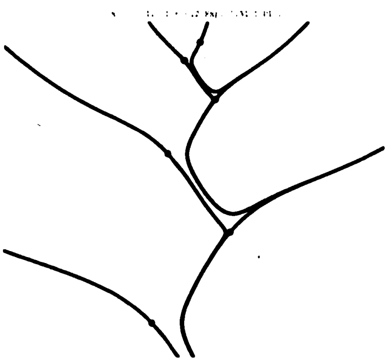

Fig.7-6

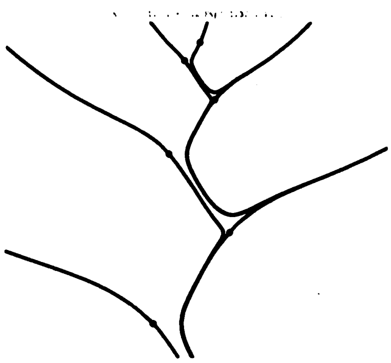

Fig.7-7

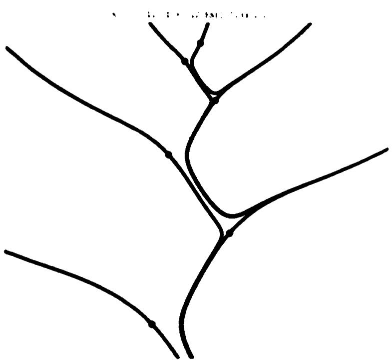

Fig.7-8

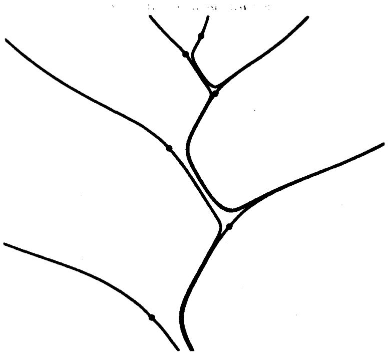
Fig.7-9

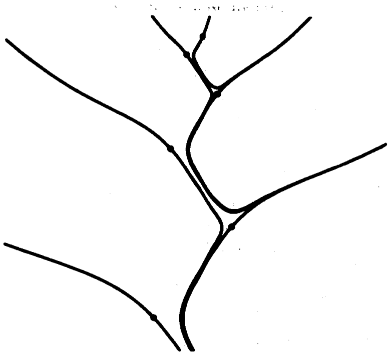
Fig.7-10


Fig.7-11


Fig.7-12


Fig.7-13

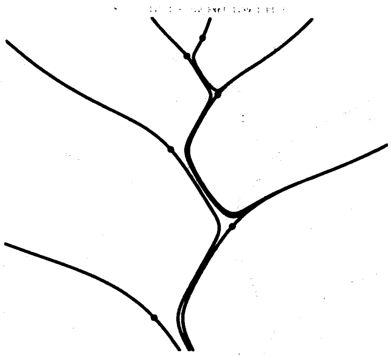
Fig.7-14

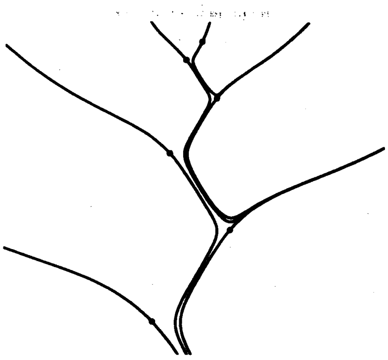
Fig.7-15

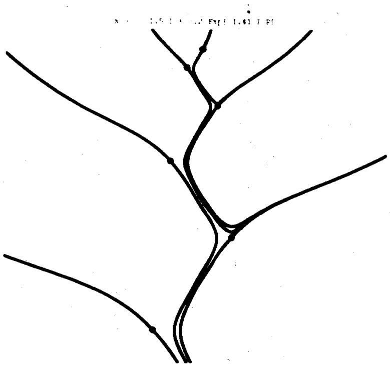
Fig.7-16


Fig.7-17

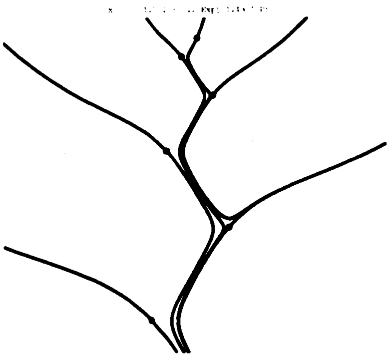

Fig.7-18

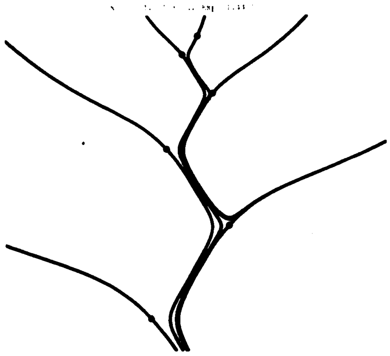

Fig.7-19

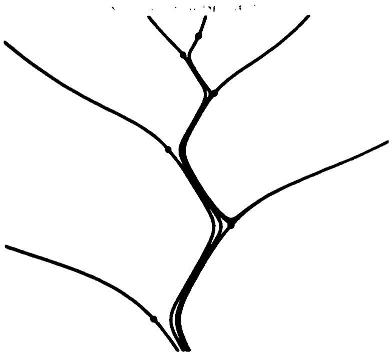

Fig.7-20

Fig.7-21

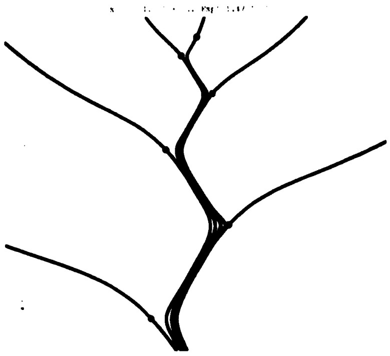

Fig.7-22

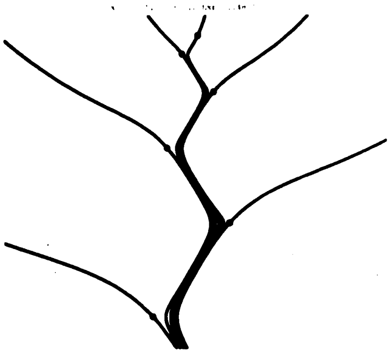

Fig.7-23

Fig.7-24

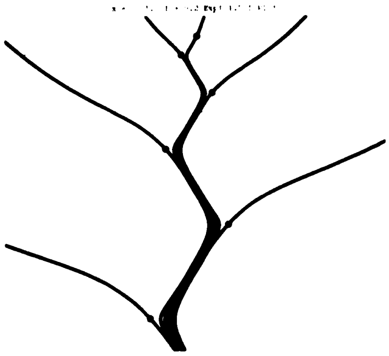

Fig.7-25

First topological change of the configuration occurs between Fig.7-5 and Fig.7-6, i.e., when we cross a Stokes curve emanating from $x = 1$. Next topological change occurs between Fig.7-17 and Fig.7-18, i.e., when we cross a new Stokes curve given by

$$\operatorname{Im} \int_{x_1}^{x} (f_n(x) - f_{-n+1}(x))dx = 0. \tag{28}$$

Another topological change of the configuration can be observed between Fig.7-22 and Fig.7-23, i.e., when we cross a new Stokes curve given by

$$\operatorname{Im} \int_{x_1}^{x} (f_n(x) - f_{-n+2}(x))dx = 0. \tag{29}$$

Surmising from these observations, we then expect that on the new Stokes curve defined by

$$\operatorname{Im} \int_{x_1}^{x} (f_n(x) - f_{-n+k}(x))dx = 0 \quad (k \geq 1) \tag{30}$$

the steepest descent path passing through the saddle point $\xi = \xi_0$ (which is located at the "top") goes down and flows into a saddle point $\xi = \xi_k$. (At the same time, two saddle points $\xi = \xi_n$ and $\xi = \xi_{-n+k}$ are connected by a steepest descent path for every n.) Thus we conclude that Stokes phenomena occur when we cross new Stokes curves designated by a solid line in Fig.3. Similar results can also be checked near $x = x_0$, the crossing point in the upper half plane.

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[AKT2] T.Aoki, T.Kawai and Y.Takei: On the exact steepest descent method – a new method for the description of Stokes curves, to appear in J. Math. Phys.

[BNR] H.L.Berk, W.M.Nevins and K.V.Roberts: New Stokes lines in WKB theory, J. Math. Phys., **23**(1982), 988-1002.

[T] Y.Takei: Integral representation for ordinary differential equations of Laplace type and exact WKB analysis, RIMS Kôkyûroku, No.1168, pp.80-92, 2000.